What Sort of Pedagogic Theory do we need for Teaching Mathematics in Higher Education?

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Introduction

In the training of new lecturers for their teaching the question arises as to the extent to which educational theory should play a role. For example, the HEA Professional Standards Framework holds as one of its principles that teaching practice should be underpinned by theory of how students learn. But there is a wide range of views amongst chalk-face mathematicians about the value and extent of such theory. So whatever theoretical underpinning of teaching and learning we provide, an important consideration is pitching the content, level, and workload at a level that won’t be counterproductive in alienating academics. This article looks at the sorts of pedagogic theory that relate to the teaching of mathematics in HE, surveys the literature and gives some practical examples of how the outcomes of such theory can have positive benefits in the classroom.

Laws, principles, theories and models for teaching and learning in Mathematics

When we, for example, talk of theories of how students learn, there can be confusion between the wide range of such theories, and between various ‘laws’ and ‘principles’ that have gained widespread currency amongst academics. For example, ‘Ausubel’s Principle’ (Ausubel, et al, 1978) on starting to teach from what the student knows has wide acceptance amongst the community, but would not be described as a ‘theory’. A good discussion of the distinctions between laws, principles, theories and models and their relations to traditional empirical scientific approaches is given by Baddeley (1991) in the context of human memory, but it is equally applicable to theories of learning in general. Briefly, Baddeley notes that laws and principles describe what we know (or think we know), whereas theories and models provide tools for learning more. Many authors on teaching mathematics base their treatments on a number of common sense principles of teaching and learning (Baumslag, 2000; Krantz, 1999). Perhaps it is not so much theories that we need, but a comprehensive collection of convincing explanations, with predictive power, for how and under what circumstances students most effectively learn mathematics. There is certainly no shortage of such material, even at the HE level, and the main difficulty is sifting out that which is of real practical benefit in the classroom. Note that theories and laws don’t have to be ‘right’ to yield practical benefits. For example, Maxwell’s idle wheels and vortices model of the (non-existent!) aether now seems at best quaint – but it gave us his equations (Cushing, 1998). So we shouldn’t be over cautious in adopting laws and theories for teaching and learning. While we would ideally want
sound empirical foundations for any theory, widespread practitioner consensus may serve just as well.

**Mathematics teaching and learning as applied Psychology**

Most educational theory has in the past evolved largely from Psychology, so first let us look at what this might tell us about teaching mathematics. Broadly, there are six theoretical approaches in Psychology (Jarvis, 2000). All of these have some application in the teaching of mathematics.

**Behavioural psychology**

Associated with the likes of Pavlov and Skinner and the idea of learning by conditioning, and very influential in education many years ago. Regarded as old fashioned now, but many of us will unashamedly use a bit of positive reinforcement to induce students to plod through the routine drills required to master routine mathematical knowledge and techniques.

**Psychodynamic psychology**

Freud, of course, with the main emphasis on the role of the unconscious in thought processes. Big on dreams, which Freud described as ‘the royal road to a knowledge of the activities of the unconscious mind’. As if to support the ‘No royal road to geometry’ quip of Euclid, Hadamard (1945) found few reports of productive mathematical work arising from dreams. However, he of course found a great deal to say about the role of the unconscious mind in mathematical creativity, famous evidence for it coming from the story of Poincare’s flash of inspiration when stepping onto an omnibus.

**Humanistic psychology**

Here, the basic assumption is that people are capable of deciding for themselves how to conduct their lives, and will, if left to themselves, do so with a motivation to fulfill their potential. The chief figures here are Carl Rogers and Abraham Maslow, both important contributors to education, but particularly the latter in the context of his hierarchy of needs as a model for motivation. There is no doubting the importance of motivation in the learning of mathematics. Indeed, Devlin (2000) argues that lack of motivation is the main reason that people don’t fulfill their real capabilities in mathematics.

**Cognitive psychology**

As the approach that focuses on how the human mind perceives, processes, stores and responds to information, this has featured largely in educational theory, mainly through its connectionist branch dealing with computer analogies to and simulations of how the brain works. It is highly scientific and therefore carries more respectability than most psychological approaches in teaching and learning. And of course, now a great deal of effort is put into computer aided learning and assessment. It has to be remembered, however, that people have always attempted to model cognitive behaviour by the latest technology (Clockwork mechanisms in Newton’s day, for example), and there is no reason to believe that the current computer analogy isn’t just another example of this. A readable survey of work in this field is that of Newell (1990), because in seeking unified theories of cognition it brings together many strands of the approach.

**Cognitive-developmental psychology**

This is of course, through the work of Piaget and Vygotsky, the branch of psychology that we must associate with teaching and learning, and its development through Skemp, Tall and others will be discussed below. This is probably the real birthplace of modern Mathematical Education.

**Social psychology**

Teaching and learning is nothing more than interaction between people, and in that sense social psychology, the psychology of human interaction, would be expected to play a major role in educational theory. In fact, at the HE level anyway, it hasn’t, to date. Maybe it will play a greater role as the full impact of widening participation and the challenges that brings are realised.

**Biological psychology**

This is perhaps the most exciting branch of psychology for the future. While we are a long way from understanding how knowledge of brain functioning, for example, might help us to understand how students learn mathematics, there is still a great deal we can learn from the rapidly emerging field of neurophysiology. For example, attention to bodily rhythms can suggest the times for getting the best out of students (Don’t try conceptually difficult material in late afternoon lectures!). For an adventurous foray into education from the neurophysiological perspective, see Geake (2003). While aimed at school teaching, this is an interesting attempt to apply current knowledge about brain functioning to educational practices.

So far as teaching in HE is concerned, outcomes from the above approaches often reach the new lecturer in generic staff development courses, or via various developments in mathematical education, so we look at these now.

**Generic learning theories**

The classic example of the sort of generic overview of the processes one goes through when engaging in learning activity is the Kolb Learning Cycle (Mors and Murray, 2005), which is often the standard model used in generic teaching and learning courses. It is an experiential cyclical process comprising concrete experience, observation
and reflection, abstract conceptualisation, and testing of concepts in new situations. It is not very sophisticated, being analogous to the way we usually approach modelling in mathematics – model the situation, abstract the mathematical model, solve the resultant mathematical problem, compare the results with experience, etc. Since most mathematicians know what a circle looks like, little of this is really challenging, and does not really help us to understand how the student struggles with ideas that we find so clear, nor helps us in helping them. A more scholarly coverage of generic theories of learning is Bigge and Shermis (1992). This is particularly useful because it considers the teaching implications of the various theories, albeit at the school level. At the HE level a good treatment is that contained in Heywood (2000). While generic, these books often draw on examples from mathematics, which probably prejudices me towards them! For a good summary of what generic learning theories tell us, from the point of view of a mathematician, see Baumslag (2000).

**Mathematical Education**

Mathematical Education has a long history of research and development by specialist mathematics educators and practicing teachers and psychologists. In 1913, Poincare gave a now famous lecture in which he tried to analyse the thought processes involved in mathematical ‘invention’ (Poincare, 1913) – which was essentially about how we come to develop new (for the learner) ideas in mathematics. He based this on introspection, about how he himself learned and developed mathematics, and also on his historical knowledge of other mathematicians. In fact, such ‘bearing of the soul’ in the way mathematicians think and learn about mathematics goes back at least to Archimedes’ ‘The Method’ (Boyer, 1991).

In an extended essay, Hadamard (Hadamard, 1945) built on Poincare’s work, summarised other’s contributions, in other subjects, as well as mathematics, and linked this with student learning of mathematics generally. Hadamard also sought to establish the results and ideas in a coherent semi-theoretical context, linking to contemporary views from the psychology of learning, and one can see in his book the seeds of many of the ideas and structures prevalent in modern mathematical education at the advanced level. Skemp (1971), coming essentially via school mathematics and psychology, built on and consolidated the ideas and established a more theoretical framework. Work since then, in what has now become a specialised area of mathematical education, is conveniently summarised in David Tall’s excellent book Advanced Mathematical Thinking (Tall, 1991). This book collates and organises the previous work and contains more recent research by the leading activists in the field. It is also an easy read and makes every effort to link with classroom practice. For further updates of the current state of the art see Holton (2001) and Sierpinska and Kilpatrick (1998).

**Practical outcomes and applications**

Busy lecturers are unlikely to engage with pedagogic theory, if it is not of real practical use in the classroom. So we will summarize some of the main points emerging from this work, which are directly related to how we operate in the classroom.

- Whatever we think of ‘learning styles, there is no doubt that we can identify different kinds of mathematical mind. For example, some of us might prefer symbolic argument while others might be guided more by visual, geometric argument. So in our own teaching, we can try to accommodate this variety and minimize any adverse effects of our own particular predilections for one approach or the other.

- There may be differences between a student’s particular concept image, our own image of the same concept and the concept definition. By being alert to such differences, we can head off possible misconceptions. For example, a good proportion of students enter university believing that \( e^{i\pi} = c^i + e^d \) and it is very difficult to change this. To shift such misconceptions, it is usually necessary to engineer some ‘cognitive conflict’ that brings the student face to face with the error. For example, one might get them to work out \( e = c^i + 1 = c^i + e^d \) etc, deducing \( 1 = 0 \).

- There seem to be distinct stages in cognitive development, from Piaget’s sensorimotor, preoperational, concrete operational, formal operational stages and on to the reflective judgement stage (Skemp 1971, Heywood 2000) of precise definitions and logic characteristic of advanced mathematics. These days, many incoming students will not have thoroughly consolidated the formal stage and this will need to be done before they can progress to more advanced mathematics. As Tall (1991) emphasises, this can require a specific curriculum organisation in the first year.

- We seem to learn by mental reconstruction of current knowledge, and students need support to assimilate and accommodate to new concepts and schema (units of memory, each of which represents all our information on one aspect of the world). The lecturer can aid this process by analysis of the curriculum, identifying where there is a need to focus on such conceptual construction tasks.

- Such conceptual steps as **generalisation** are difficult for students, particularly first years, and the teacher can help them by providing appropriate examples, testing specifically the generalisations they are supposed to have made, etc. Again **abstraction** is an even more difficult process in which many students need help. It is not so much that the students cannot generalise and abstract (after all, there are few more abstract notions than number itself), it is more that we often overestimate how quickly they can do it and do not realise the amount and nature of the support they need.
• The way we teach mathematics is of course not always the way we do mathematics, and the students may benefit from being exposed to more of the latter. We can display our mathematical thinking more openly and rely less on inexorable sequential logic. We can make explanations ‘sensible’ as much as ‘logical’. We can demonstrate the dual use of rigour and intuition and encourage students to feel safe about guessing (and checking), and less afraid of being wrong. But then they also need to understand the nature of rigour and be schooled in the intense self critical and precise thinking that it entails.

• We often find students are not very proficient at solving problems of any real substance. In fact, this has always been an issue, and authors such as Polya (1945) have long ago attempted to address this. But for today’s students, much of the well-intentioned advice is insufficiently structured or detailed. For example, as noted by Tall (1991), Polya’s modelling framework of: understand the problem, devise a plan, carry out the plan, and look back at the work is not much help for the student who doesn’t know how to devise a plan. Authors such as Mason et al (1982) and Schoenfeld (1985) have developed more practical and detailed advice on how to tackle problems and provide many useful ideas for the lecturer.

The experienced and thoughtful teacher will appreciate such points as above, and will have developed their own approaches to them. However, they are not always obvious to the less experienced lecturer, and even the experienced lecturer might be grateful for ideas that abound in the literature for dealing with them. And none of it comes from any one theory, but an eclectic mix of different theories, different approaches and sources. A more detailed coverage of ‘tactics’, based on sound educational theory, can be found in Mason (2002). Another useful resource is the MSOR Network database of reviewed educational research in MSOR (2003). Also there is much to learn about learning from other subject areas, particularly the sciences and engineering. Useful recent references are Reid (2003) and Palmer and Reid (2003).

Conclusion
In surveying the work on ‘theories’ of learning that might be relevant in mathematics teaching in HE, we find a great deal that is of practical use. The question now is how to articulate and disseminate this for the benefit of students and lecturers. In doing this, we feel that we should be guided by four main principles. The presentation of pedagogic theory should:

• have a coherent structure based on robust principles;
• be based as much as possible on empirical evidence, which may include practitioner consensus;
• lead to tangible practical benefits in the classroom for students’ learning;
• be sufficiently convincing and useful to be embraced by practitioners.

References
16. Maths, Stats, & OR Network Resources (2003) A Database of reviewed educational research in MSOR (http://ltsn.mathstore.gla.ac.uk/resourcecollection/)


